

# On the convergence of the affine hull of the Chvátal-Gomory closures\*

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## Abstract

Given an integral polyhedron  $P \subseteq \mathbb{R}^n$  and a rational polyhedron  $Q \subseteq \mathbb{R}^n$  containing the same integer points as  $P$ , we investigate how many iterations of the Chvátal-Gomory closure operator have to be performed on  $Q$  to obtain a polyhedron contained in the affine hull of  $P$ . We show that if  $P$  contains an integer point in its relative interior, then such a number of iterations can be bounded by a function depending only on  $n$ . On the other hand, we prove that if  $P$  is not full-dimensional and does not contain any integer point in its relative interior, then no finite bound on the number of iterations exists.

**Key words.** affine hull, Chvátal-Gomory closure, Chvátal rank, cutting plane, integral polyhedron

**AMS subject classification.** 90C10, 52B20, 52C07

## 1 Introduction

The *integer hull*  $Q_I$  of a polyhedron  $Q \subseteq \mathbb{R}^n$  is defined by  $Q_I := \text{conv}(Q \cap \mathbb{Z}^n)$ , where “conv” denotes the convex hull operator. If  $Q$  is a rational polyhedron, then  $Q_I$  is a polyhedron [14] (see also [21, §16.2]). A polyhedron  $Q$  is *integral* if  $Q = Q_I$ . Given an integral polyhedron  $P \subseteq \mathbb{R}^n$ , a *relaxation* of  $P$  is a rational polyhedron  $Q \subseteq \mathbb{R}^n$  such that  $Q \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n$ . An inequality  $cx \leq \lfloor \delta \rfloor$  is a *Chvátal-Gomory inequality* (*CG inequality* for short) for a polyhedron  $Q \subseteq \mathbb{R}^n$  if  $c$  is an integer vector and  $cx \leq \delta$  is valid for  $Q$ . Note that  $cx \leq \lfloor \delta \rfloor$  is a valid inequality for  $Q \cap \mathbb{Z}^n$  and thus also for  $Q_I$ . The *CG closure*  $Q'$  of  $Q$  is the set of points in  $Q$  that satisfy all the CG inequalities for  $Q$ . If  $Q$  is a rational polyhedron, then  $Q'$  is again a rational polyhedron [20, Theorem 1]. For  $k \in \mathbb{N}$ ,

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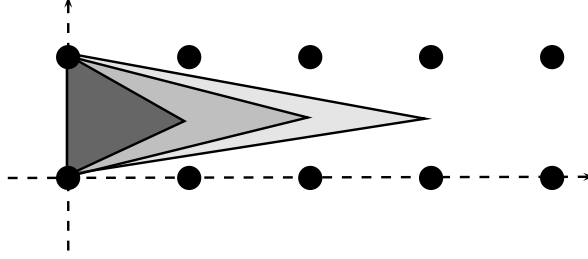


Figure 1: In increasingly lighter shades of grey, polytopes  $Q_1$ ,  $Q_2$ , and  $Q_3$ .

the  $k$ -th CG closure  $Q^{(k)}$  of  $Q$  is defined iteratively as  $Q^{(k)} := (Q^{(k-1)})'$ , with  $Q^{(0)} := Q$ . If  $Q$  is a rational polyhedron, then there exists a nonnegative integer  $p$  such that  $Q^{(p)} = Q_I$  [20, Theorem 2]. In other words, the sequence of polyhedra  $(Q^{(k)})_{k \in \mathbb{N}}$  finitely converges to  $Q_I$ . The minimum nonnegative integer  $p$  for which  $Q^{(p)} = Q_I$ , called the CG rank of  $Q$  and denoted by  $r(Q)$ , can be viewed as the rate of finite convergence of the sequence  $(Q^{(k)})_{k \in \mathbb{N}}$  to  $Q_I$ .

It is well known that, already in dimension 2, the CG rank of rational polyhedra can be arbitrarily high. In fact, for  $t \in \mathbb{N}$ , consider the polyhedron  $Q_t := \text{conv}(\{(0,0), (0,1), (t, 1/2)\})$  (see Figure 1). Note that  $(Q_t)_I = \text{conv}\{(0,0), (0,1)\}$  for all  $t \in \mathbb{N}$ . It is folklore that  $Q_t \subseteq Q'_{t+1}$ , hence by induction  $r(Q_t) \geq t$ . This example shows actually something stronger: it can take arbitrarily many rounds of the CG closure already for a polyhedron to be contained in the affine hull of its integer points (the *affine hull* of a set  $S \subseteq \mathbb{R}^n$  is the smallest affine set containing  $S$ , and throughout the paper it will be denoted by  $\text{aff}(S)$ ). The purpose of this paper is to provide a systematic study of this phenomenon. More precisely, we consider the following question: given an integral polyhedron  $P \subseteq \mathbb{R}^n$ , is there an integer  $p$  such that, for each relaxation  $Q$  of  $P$ , one has  $Q^{(p)} \subseteq \text{aff}(P)$ ? The example from Figure 1 shows that such a  $p$  does not always exist. However, as our main result, we prove that if  $P$  contains an integer point in its relative interior, then such a  $p$  exists, and indeed it depends only on  $n$ .

**Theorem 1.** *There exists a function  $\beta : \mathbb{N} \rightarrow \mathbb{N}$  such that, for each integral polyhedron  $P \subseteq \mathbb{R}^n$  containing an integer point in its relative interior and each relaxation  $Q$  of  $P$ ,  $Q^{(\beta(n))}$  is contained in  $\text{aff}(P)$ .*

The above theorem can be interpreted as follows: for the family of rational polyhedra  $Q \subseteq \mathbb{R}^n$  such that  $Q_I$  contains an integer point in its relative interior, there is a global upper bound on the rate of finite convergence of the sequence  $(\text{aff}(Q^{(k)}))_{k \in \mathbb{N}}$  to  $\text{aff}(Q_I)$ , where this global upper bound depends on  $n$  only. Theorem 1 is proved in Section 3, after some preliminaries in Section 2.

We complement Theorem 1 by generalizing the construction from Figure 1 to any dimension. More precisely, we show in Section 4 that if  $P \subseteq \mathbb{R}^n$  is not full-dimensional (i.e., its dimension is smaller than  $n$ ) and does not contain any integer point in its relative interior, then there is no  $p$  such that  $Q^{(p)} \subseteq \text{aff}(P)$  for every relaxation  $Q$  of  $P$ .

**Theorem 2.** *If a non-full-dimensional integral polyhedron  $P \subseteq \mathbb{R}^n$  does not contain any integer point in its relative interior, then for each  $k \in \mathbb{N}$  there exists a relaxation  $Q$  of  $P$  such that  $Q^{(k)}$  is not contained in  $\text{aff}(P)$ .*

Let us remark that Theorems 1 and 2 show a qualitative difference between those non-full-dimensional rational polyhedra that have an integer point in the relative interior of their integer hull, and those which do not. However, we prove that even for the former class, we cannot hope in general to converge to the affine hull after a small number of CG closures: in Section 5 we show the best choice for the function  $\beta$  from Theorem 1 is doubly exponential in  $n$  ( $\beta(n) = 2^{2^{\Theta(n)}}$ ).

We conclude in Section 6, where we prove the NP-completeness of a decision problem, denoted **AFF-IHULL**, which is closely related to the questions considered above: given  $x \in \mathbb{Q}^n$  and a rational polyhedron  $Q \subseteq \mathbb{R}^n$  (described by a system of linear inequalities with rational coefficients), decide whether  $x \in \text{aff}(Q_I)$ .

To the best of our knowledge, the questions we investigate in this paper have not been addressed before. In fact, most works on the CG rank of polyhedra aim at bounding the rank of rational polyhedra contained in the 0-1 cube (see, e.g., [7, 16, 17, 18]).

## 2 Tools

In this section we provide some definitions and auxiliary results that will be used in the proofs of the main theorems. For standard background material on convex sets, polyhedra, geometry of numbers and integer optimization, we refer the reader to the monographs [4, 10, 19, 21].

Throughout the paper,  $n$  will be a positive integer denoting the dimension of the ambient space. Given a closed convex set  $C \subseteq \mathbb{R}^n$ , we denote by  $\text{int}(C)$  the interior of  $C$ , by  $\text{relint}(C)$  the relative interior of  $C$ , and by  $\text{rec}(C)$  the recession cone of  $C$ . We say that  $C$  is *lattice-free* if  $\text{int}(C) \cap \mathbb{Z}^n = \emptyset$ , and *relatively lattice-free* if  $\text{relint}(C) \cap \mathbb{Z}^n = \emptyset$ . Note that if  $C$  is not lattice-free, then it is full-dimensional. A *convex body* is a closed, convex, bounded subset of  $\mathbb{R}^n$  with non-empty interior. A set  $C \subseteq \mathbb{R}^n$  is *centrally symmetric* with respect to a given point  $x \in C$  (or centered at  $x$ ) when, for every  $y \in \mathbb{R}^n$ , one has  $x + y \in C$  if and only if  $x - y \in C$ . When talking about distance and norm, we always refer to the Euclidean distance and the Euclidean norm; in particular, we denote the latter using the standard notation  $\|\cdot\|$ . For  $d \in \{1, \dots, n\}$ , when referring to the volume of a  $d$ -dimensional convex set  $C \subseteq \mathbb{R}^n$ , denoted  $\text{vol}(C)$ , we shall always mean its  $d$ -dimensional volume, that is, the Lebesgue measure with respect to the affine subspace  $\text{aff}(C)$  of the Euclidean space  $\mathbb{R}^n$ . For  $i \in \mathbb{N}$ , we write  $e^i$  to denote the unit vector of suitable dimension with 1 in its  $i$ -th entry and 0 elsewhere. The all-zero vector of appropriate dimension is denoted by  $\mathbf{0}$ .

## 2.1 Integer points in convex sets

We will make use of Minkowski's Convex Body Theorem, which we state below (see, e.g., [4, Chapter 7, §3]).

**Theorem 3** (Minkowski's Convex Body Theorem). *Let  $C \subseteq \mathbb{R}^n$  be a centrally symmetric convex body centered at the origin. If  $\text{vol}(C) \geq 2^n$ , then  $C$  contains a non-zero integer point.*

The *lattice width* of a closed convex set  $C \subseteq \mathbb{R}^n$  is defined (for the integer lattice  $\mathbb{Z}^n$ ) by

$$w(C) := \inf_{c \in \mathbb{Z}^n \setminus \{0\}} \left\{ \sup_{x \in C} cx - \inf_{x \in C} cx \right\}.$$

If  $C$  is full-dimensional and  $w(C) < +\infty$ , then there exists a non-zero integer vector  $c$  for which

$$w(C) = \max_{x \in C} cx - \min_{x \in C} cx.$$

The following theorem is due to Khintchine [12] (see also [3, 11] and [4, Chapter 7, §8] for related results and improvements).

**Theorem 4** (Flatness Theorem). *For every convex body  $C \subseteq \mathbb{R}^n$  with  $C \cap \mathbb{Z}^n = \emptyset$ , one has  $w(C) \leq \omega(n)$ , where  $\omega$  is a function depending on  $n$  only.*

The following is a simple corollary.

**Corollary 5.** *For every  $k \in \mathbb{N}$  and every convex body  $C \subseteq \mathbb{R}^n$  with  $|C \cap k\mathbb{Z}^n| = 1$ , one has  $w(C) \leq \omega(n, k)$ , where  $\omega$  is a function depending on  $n$  and  $k$  only.*

*Proof.* Wlog we can assume that the only point in  $C \cap k\mathbb{Z}^n$  is the vector  $(k, \dots, k)$ . This implies that  $\frac{1}{2k}C$  contains no integer point. By Theorem 4,  $w(\frac{1}{2k}C)$  is bounded by some function depending only on the dimension  $n$ . As  $w(C) = 2k \cdot w(\frac{1}{2k}C)$ ,  $w(C)$  is bounded by some function depending only on  $n$  and  $k$ .  $\square$

The above proof shows that one can set  $\omega(n, k) = 2k\omega(n)$ , with  $\omega(n)$  as in Theorem 4. Since one can choose  $\omega(n) = O(n^{3/2})$  (see [3, Proposition 2.3 and Theorem 2.4]), we obtain  $\omega(n, k) = O(kn^{3/2})$  (as both  $n, k \rightarrow \infty$ ). We remark that a slightly worse bound of  $O(kn^2)$  follows from [11, Theorem (4.1)].

We will need the result of Corollary 5 for (possibly unbounded) rational polyhedra rather than convex bodies (which are bounded by definition). Thus we will make use of the following result.

**Corollary 6.** *For every  $k \in \mathbb{N}$  and every rational polyhedron  $Q \subseteq \mathbb{R}^n$  with  $|Q \cap k\mathbb{Z}^n| = 1$ , one has  $w(Q) \leq \omega(n, k)$ , where  $\omega$  is a function depending on  $n$  and  $k$  only.*

*Proof.* Since  $|Q \cap k\mathbb{Z}^n| = 1$ , we have  $|\frac{1}{k}Q \cap \mathbb{Z}^n| = 1$ . So  $\frac{1}{k}Q$  is a rational polyhedron containing precisely one integer point, which we denote by  $z$ . If  $\frac{1}{k}Q$  were unbounded, then the sum of  $z$  and any of the infinitely many integer vectors in the recession cone of  $\frac{1}{k}Q$  would be in  $\frac{1}{k}Q$ , contradicting  $|\frac{1}{k}Q \cap \mathbb{Z}^n| = 1$ . Hence  $\frac{1}{k}Q$  is bounded, which implies that  $Q$  is bounded as well. Then the assumptions of Corollary 5 are fulfilled for  $Q$  and thus  $w(Q) \leq \omega(n, k)$ .  $\square$

Corollary 6 can also be proven as a consequence of Corollary 5 and an observation of Eisenbrand and Shmonin [8], who noticed that the Flatness Theorem remains true if  $C$  is assumed to be a rational polyhedron instead of a convex body (see the discussion after Theorem 2.1 in [8]).

## 2.2 Upper bounds on the CG rank

The next lemma provides an important property of CG inequalities; for its proof see, e.g., [21, page 340].

**Lemma 7.** *Let  $Q \subseteq \mathbb{R}^n$  be a rational polyhedron and  $F$  be a face of  $Q$ . Then for each  $t \in \mathbb{N}$ ,  $F^{(t)} = F \cap Q^{(t)}$ .*

Using the Flatness Theorem and the previous lemma, one can prove the following result [6, Theorem 1'] (see also [21, Theorem 23.3]).

**Lemma 8.** *The CG rank of every rational polyhedron  $Q \subseteq \mathbb{R}^n$  with  $Q \cap \mathbb{Z}^n = \emptyset$  is at most  $\varphi(n)$ , where  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  is a function depending on  $n$  only.*

Wlog we assume that the values  $\varphi(n)$  from Lemma 8 are non-decreasing in  $n$ . We use the previous results to show the following upper bound on the CG rank.

**Lemma 9.** *For every rational polyhedron  $Q \subseteq \mathbb{R}^n$  and every  $c \in \mathbb{Z}^n$  and  $\delta, \delta' \in \mathbb{R}$  (with  $\delta' \geq \delta$ ) such that  $cx \leq \delta$  is valid for  $Q_I$  and  $cx \leq \delta'$  is valid for  $Q$ , the inequality  $cx \leq \delta$  is valid for  $Q^{(p+1)}$ , where  $p = (\lfloor \delta' \rfloor - \lfloor \delta \rfloor)\theta(n)$  and  $\theta : \mathbb{N} \rightarrow \mathbb{N}$  is a function depending on  $n$  only.*

*Proof.* We assume  $n \geq 2$ , otherwise the statement is trivially verified. We show that, for each integer  $0 \leq k \leq \lfloor \delta' \rfloor - \lfloor \delta \rfloor$ , the inequality  $cx \leq \delta' - k$  is valid for  $Q^{(k\varphi(n-1)+k+1)}$ , with  $\varphi$  being the function from Lemma 8. Note that  $Q^{(1)} \subseteq \{x \in \mathbb{R}^n : cx \leq \lfloor \delta' \rfloor\}$ . If  $\lfloor \delta' \rfloor \leq \delta$ , the proof is complete. Otherwise, the hyperplane  $\{x \in \mathbb{R}^n : cx = \lfloor \delta' \rfloor\}$  induces a face  $F$  of  $Q^{(1)}$  of dimension at most  $n - 1$  containing no integer points. From Lemmas 7 and 8 we conclude that  $\{x \in \mathbb{R}^n : cx = \lfloor \delta' \rfloor\} \cap Q^{(\varphi(n-1)+1)} = \emptyset$ , and consequently  $Q^{(\varphi(n-1)+2)} \subseteq \{x \in \mathbb{R}^n : cx \leq \lfloor \delta' \rfloor - 1\}$ . By iterating this argument, we obtain  $Q^{(k\varphi(n-1)+k+1)} \subseteq \{x \in \mathbb{R}^n : cx \leq \lfloor \delta' \rfloor - k\}$  for  $0 \leq k \leq \lfloor \delta' \rfloor - \lfloor \delta \rfloor$ . In particular, when  $k = \lfloor \delta' \rfloor - \lfloor \delta \rfloor$  one has that  $cx \leq \lfloor \delta \rfloor$  is valid for  $Q^{((\lfloor \delta' \rfloor - \lfloor \delta \rfloor)\theta(n)+1)}$  with  $\theta(n) := \varphi(n - 1) + 1$ .  $\square$

## 2.3 Unimodular transformations

A *unimodular transformation*  $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$  maps a point  $x \in \mathbb{R}^n$  to  $u(x) = Ux + v$ , where  $U$  is a unimodular  $n \times n$  matrix (i.e., a square integer matrix with  $|\det(U)| = 1$ ) and  $v \in \mathbb{Z}^n$ . It is well-known (see, e.g., [21, Theorem 4.3]) that a non-singular matrix  $U$  is unimodular if and only if so is  $U^{-1}$ . Furthermore, a unimodular transformation is a bijection of both  $\mathbb{R}^n$  and  $\mathbb{Z}^n$  that preserves  $n$ -dimensional volumes. Moreover, the following holds (see [7, Lemma 4.3]).

**Lemma 10.** *If  $Q \subseteq \mathbb{R}^n$  is a rational polyhedron,  $t \in \mathbb{N}$  and  $u : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a unimodular transformation, then  $u(Q^{(t)}) = u(Q)^{(t)}$ . In particular, the CG ranks of  $Q$  and  $u(Q)$  coincide.*

Because of the previous lemma, when investigating the CG rank of a  $d$ -dimensional rational polyhedron  $Q \subseteq \mathbb{R}^n$  with  $Q \cap \mathbb{Z}^n \neq \emptyset$ , we can apply a suitable unimodular transformation and assume that  $Q$  is contained in the rational subspace  $\mathbb{R}^d \times \{0\}^{n-d} = \{x \in \mathbb{R}^n : x_{d+1} = x_{d+2} = \dots = x_n = 0\}$ .

## 2.4 Centrally symmetric convex bodies in integral polyhedra

Given a convex body  $C \subseteq \mathbb{R}^n$  and a point  $x \in \text{int}(C)$ , we define the *coefficient of asymmetry of  $C$  with respect to  $x$*  to be the value

$$\gamma(C, x) := \max_{z \in \mathbb{R}^n : \|z\|=1} \frac{\max\{\lambda : x + \lambda z \in C\}}{\max\{\lambda : x - \lambda z \in C\}}.$$

Note that  $\gamma(C, x) \geq 1$ , and  $\gamma(C, x) = 1$  if and only if  $C$  is centrally symmetric with respect to  $x$ . Furthermore, assuming for the sake of simplicity that  $x$  is the origin,  $\gamma(C, x)$  is the smallest number  $\gamma$  for which  $\frac{1}{\gamma}C \subseteq -C$ .

The following result was proved by Pikhurko [15, Theorem 4].

**Theorem 11.** *There exists a function  $\sigma : \mathbb{N} \rightarrow ]0, +\infty[$  such that every non-lattice-free integral polytope  $P \subseteq \mathbb{R}^n$  contains a point  $x \in \text{int}(P) \cap \mathbb{Z}^n$  satisfying  $\gamma(P, x) \leq \sigma(n)$ .*

The main result of this section is the following.

**Theorem 12.** *There exists a function  $\nu : \mathbb{N} \rightarrow ]0, +\infty[$  such that every non-lattice-free integral polyhedron  $P \subseteq \mathbb{R}^n$  contains a centrally symmetric convex body of volume  $\nu(n)$ , whose only integer point is its center.*

*Proof.* First of all we show that we can assume wlog that  $P$  is bounded. This can be proven by using Steinitz' theorem (see, e.g., [19, Theorem 1.3.10]), which is as follows: given a set  $T \subseteq \mathbb{R}^n$  and a point  $z \in \text{int}(\text{conv}(T))$ , there exists a subset  $T' \subseteq T$  with  $|T'| \leq 2n$  such that  $z \in \text{int}(\text{conv}(T'))$ . Now, let  $P$  be a non-lattice-free integral polyhedron and take  $z \in \text{int}(P) \cap \mathbb{Z}^n$ . Apply Steinitz' theorem with  $T = P \cap \mathbb{Z}^n$  and note that  $\text{conv}(T')$  is a non-lattice-free integral polytope contained in  $P$ . Therefore, replacing  $P$  with  $\text{conv}(T')$ , we may assume that  $P$  is a polytope.

By Theorem 11, there exists  $x \in \text{int}(P) \cap \mathbb{Z}^n$  such that  $\gamma(P, x) \leq \sigma(n)$ . To simplify notation, we write  $\gamma$  instead of  $\gamma(P, x)$  and assume wlog  $x = \mathbf{0}$ . Define  $Q = \text{conv}(P \cup -P)$ . Note that  $Q$  is a centrally symmetric full-dimensional integral polytope containing the origin in its interior.

We claim that there exists a centrally symmetric full-dimensional integral polytope  $\bar{Q} \subseteq Q$  such that the origin is the unique integer point in  $\text{int}(\bar{Q})$ . If the origin is the unique integer point in  $\text{int}(Q)$ , then we can take  $\bar{Q} = Q$ . Therefore we assume that there exists  $y \in \text{int}(Q) \cap \mathbb{Z}^n$  with  $y \neq \mathbf{0}$ . Let  $\{y, v^1, \dots, v^{n-1}\}$  be a basis of  $\mathbb{R}^n$ , where  $v^1, \dots, v^{n-1}$  are vertices of  $Q$  (the existence of such a basis follows from the full-dimensionality of  $Q$ ). Then the polytope obtained

as the convex hull of the points  $\pm y, \pm v^1, \dots, \pm v^{n-1}$  is a centrally symmetric full-dimensional integral polytope contained in  $Q$ . Since  $y$  does not lie in the interior of this polytope, we can iterate this procedure finitely many times until we obtain a centrally symmetric full-dimensional integral polytope  $\bar{Q} \subseteq Q$  such that the origin is the unique integer point in  $\text{int}(\bar{Q})$ .

If we define  $S = \frac{1}{\gamma}\bar{Q}$ , we have  $S \subseteq \text{conv}\left(\frac{1}{\gamma}P \cup -\frac{1}{\gamma}P\right)$ . By the definition of the coefficient of asymmetry,  $\pm\frac{1}{\gamma}P \subseteq P$ , thus  $S \subseteq P$ . Furthermore,

$$\text{vol}(S) = \frac{1}{\gamma^n} \text{vol}(\bar{Q}) \geq \frac{2^n}{\gamma^n n!},$$

where the last inequality holds because  $\bar{Q}$  is the union of  $2^n$  full-dimensional integral simplices with disjoint interiors. Since  $\gamma \leq \sigma(n)$ , we obtain the bound  $\text{vol}(S) \geq 2^n/(\sigma(n)^n n!)$ , which depends only on  $n$ . Finally, the origin is the unique integer point in  $S$ , as  $S \subseteq \bar{Q}$ .  $\square$

Wlog we assume that the values  $\nu(n)$  from Theorem 12 are non-increasing in  $n$ .

We remark that Theorem 12 does not hold if, instead of looking for *any* centrally symmetric convex body, we ask for the existence of a *specific* full-dimensional centrally symmetric convex body  $S$  and a number  $t > 0$ , both depending on  $n$  only, such that  $tS$  is contained in  $P$  (up to a translation by an integer vector) and has its center as its unique integer point. This is shown by the following simple example. Let  $(P_k)_{k \in \mathbb{N}}$  be the sequence of parallelograms in  $\mathbb{R}^2$  given by  $P_k := \text{conv}(\{\pm(k, 1), \pm(1, 0)\})$ . The only integer point in  $\text{int}(P_k)$  is the origin. One readily verifies that the distance between the origin and the boundary of the parallelogram cannot be lower-bounded by a constant. Then, if  $S$  and  $t$  are fixed as above, for  $k$  large enough  $tS$  is not contained in  $P_k$ .

### 3 Proof of Theorem 1

We show Theorem 1 by induction on the codimension  $n - d$  of  $P$ , the case  $d = n$  being trivial. Hence we fix integers  $d, n$  with  $0 \leq d < n$ .

Let  $P \subseteq \mathbb{R}^n$  be a  $d$ -dimensional integral polyhedron that is not relatively-lattice-free. Up to unimodular transformations, we may assume that  $P \subseteq \mathbb{R}^d \times \{0\}^{n-d}$  and  $\mathbf{0} \in \text{relint}(P)$ , since  $P$  is a rational polyhedron with at least one integer point in its relative interior. Assuming for the moment  $d > 0$ , there exists a  $d$ -dimensional centrally symmetric compact convex set  $S$  contained in  $P$ , centered at some integer point of  $P$ , whose only integer point is its center and whose volume is  $\nu(d)$  (see Theorem 12). Let  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-d}$  be the orthogonal projection onto the space of the last  $n - d$  components. Note that  $\pi(P) = \{\mathbf{0}\}$ . Also, let  $k = \lceil 2^{n-1} n / \nu(n) \rceil$ . Note that  $k$  depends only on  $n$ .

**Claim.**  $\pi(Q) \cap k\mathbb{Z}^{n-d} = \{\mathbf{0}\}$  for every relaxation  $Q$  of  $P$ .

*Proof.* After translating  $P$  by an integer vector, we may assume that  $S$  is centered at  $\mathbf{0}$ . Assume by contradiction that the claim is false, i.e., there exist

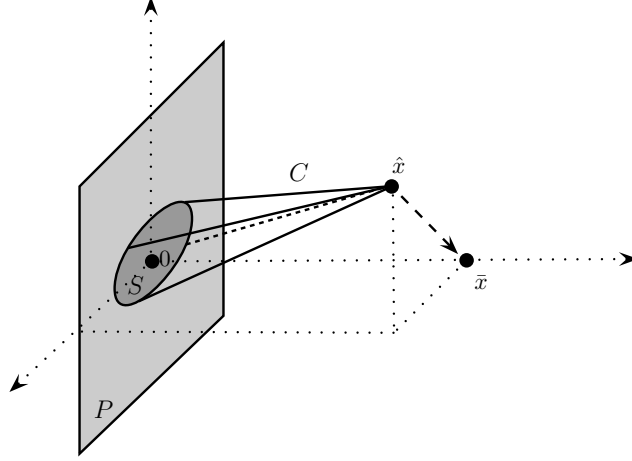


Figure 2: Illustration of the proof of the claim for the case  $d = 2$  and  $n = 3$ .

a relaxation  $Q$  of  $P$  and a point  $\bar{x} \in \pi(Q) \cap k\mathbb{Z}^{n-d} \setminus \{\mathbf{0}\}$  (see Figure 2). Let  $\hat{x} \in Q$  be such that  $\pi(\hat{x}) = \bar{x}$ . We can assume wlog that the integer vector  $\bar{x}/k$  is a primitive vector, i.e, there are no integer points in the open segment  $]0, \bar{x}/k[$ . Then, by applying a suitable linear unimodular transformation in the space  $\mathbb{R}^n$  which keeps the first  $d$  components (and thus also  $P$ ) unchanged, we may assume that  $\bar{x}/k = e^1 \in \mathbb{R}^{n-d}$ .

We define the  $(d+1)$ -dimensional compact convex set  $C := \text{conv}(S \cup \{\hat{x}\}) \subseteq \mathbb{R}^n$ . Note that  $C \subseteq Q$ , as both  $S$  and  $\hat{x}$  are contained in  $Q$ . Let  $\bar{C}$  be the  $(d+1)$ -dimensional centrally symmetric compact convex set defined as  $\bar{C} = C \cup -C$ . Note that  $\bar{C}$  lies in the space of the first  $d+1$  components. The volume of  $\bar{C}$  can be bounded as follows:

$$\text{vol}(\bar{C}) \geq \frac{2k\nu(d)}{d+1} \geq \frac{2k\nu(n)}{n} \geq 2^n,$$

where the last inequality follows from the choice of  $k$ . By Minkowski's Convex Body Theorem (Theorem 3),  $\bar{C}$  contains two non-zero integer points  $z^1$  and  $z^2$  with  $z^1 = -z^2$ . Since  $S \cap \mathbb{Z}^n = \{\mathbf{0}\}$  and  $\bar{C} \cap \text{aff}(P) = S$ ,  $z^1$  and  $z^2$  do not lie in  $\text{aff}(P)$  and so they are not contained in  $P$ . By symmetry of  $\bar{C}$ , we can assume that  $z^1$  lies in  $C$ . As  $C \subseteq Q$ ,  $z^1$  is an integer point contained in  $Q$  and not in  $P$ , contradicting the fact that  $Q$  is a relaxation of  $P$ .  $\diamond$

Note that though the above arguments cannot be used when  $d = 0$ , the claim also holds for  $d = 0$ , as in this case  $P = \{\mathbf{0}\}$ .

By Corollary 6 and the above claim, for each relaxation  $Q$  of  $P$  there exists a number  $\psi(n)$  depending only on  $n$  such that  $w(\pi(Q)) \leq \psi(n)$ . Now fix a relaxation  $Q$  and let  $c \in \mathbb{Z}^{n-d} \setminus \{\mathbf{0}\}$  be such that

$$w(\pi(Q)) = \max_{x \in \pi(Q)} cx - \min_{x \in \pi(Q)} cx.$$

Define  $c' = (0, \dots, 0, c) \in \mathbb{Z}^n$ . Note that

$$\max_{x \in Q} c'x = \max_{x \in \pi(Q)} cx \quad \text{and} \quad \min_{x \in Q} c'x = \min_{x \in \pi(Q)} cx,$$



and  $c'$  is orthogonal to  $\text{aff}(P)$ , thus  $c'x = 0$  for every  $x \in P$ . It follows by Lemma 9 that the equation  $c'x = 0$  is valid for  $Q^{(p')}$ , where  $p' = \psi(n)\theta(n) + 1$ . Therefore, the value  $p'$  depends on  $n$  only. By construction,  $c'$  is a primitive vector. Hence, up to linear unimodular transformations, we may assume that  $c' = e^n$ . Thus  $P$  can be written as  $P = \bar{P} \times \{0\}$  for an integral polyhedron  $\bar{P} \subseteq \mathbb{R}^{n-1}$ . Similarly,  $Q^{(p')}$  can be written as  $Q^{(p')} = \bar{Q} \times \{0\}$ , for a rational polyhedron  $\bar{Q} \subseteq \mathbb{R}^{n-1}$ . Since the codimension  $n - 1 - d$  of  $\bar{P} \subseteq \mathbb{R}^{n-1}$  is smaller than the codimension  $n - d$  of  $P \subseteq \mathbb{R}^n$ , it follows by induction that there exists  $p \in \mathbb{N}$ , which again only depends on  $n$ , such that  $Q^{(p+p')} \subseteq \text{aff}(P)$ . This concludes the proof of Theorem 1.

## 4 Proof of Theorem 2

A lemma of Chvátal, Cook, and Hartmann [5, Lemma 2.1] gives sufficient conditions for a sequence of points to be in successive Chvátal closures of a rational polyhedron. The one we provide next is a less general, albeit sufficient for our needs, version of their original lemma.

**Lemma 13.** *Let  $Q \subseteq \mathbb{R}^n$  be a rational polyhedron,  $x \in Q$ ,  $v \in \mathbb{R}^n$  and  $p \in \mathbb{N}$ . For  $j \in \{1, \dots, p\}$ , define  $x^j := x - j \cdot v$ . Assume that, for all  $j \in \{1, \dots, p\}$  and every inequality  $cx \leq \delta$  valid for  $Q_I$  with  $c \in \mathbb{Z}^n$  and  $cv < 1$ , one has  $cx^j \leq \delta$ . Then  $x^j \in Q^{(j)}$  for all  $j \in \{1, \dots, p\}$ .*

We now prove Theorem 2. Let  $P \subseteq \mathbb{R}^n$  be a relatively lattice-free integral polyhedron  $P$  of dimension  $d < n$ . Up to unimodular transformations, we can assume that  $P \subseteq \mathbb{R}^{n-1} \times \{0\}$ . Let  $\bar{x}$  be a point in the relative interior of  $P$ . Fix  $k \in \mathbb{N}$  and define  $Q$  as the (topological) closure of  $\text{conv}(P \cup \{\bar{x} + (k+1)e^n\})$ . Since  $Q$  is a rational polyhedron (see [2]),  $Q$  is a relaxation of  $P$ . We now argue that  $Q^{(k)}$  is not contained in  $\text{aff}(P)$ . We apply Lemma 13 with  $x = \bar{x} + (k+1)e^n$ ,  $v = e^n$  and  $p = k+1$ . Let  $cx \leq \delta$  be an inequality with  $c \in \mathbb{Z}^n$ . If  $cx \leq \delta$  is valid for  $Q_I = P$  and satisfies  $cv = c_n < 1$ , then  $c_n \leq 0$ , since both  $c$  and  $v$  are integer vectors. Then, for  $1 \leq j \leq k+1$ ,

$$cx^j = \sum_{i=1}^{n-1} c_i x_i^j + (k+1-j)c_n = c\bar{x} + (k+1-j)c_n \leq \delta.$$

Lemma 13 gives in particular  $x^k \in Q^{(k)}$ . Since  $x^k \notin \text{aff}(P)$ , this concludes the proof of Theorem 2.

## 5 Asymptotic behavior of $\beta(n)$

In this section we prove that if in Theorem 1 one chooses the minimum  $\beta(n)$  for  $n \in \mathbb{N}$ , then  $\beta(n) = 2^{2^{\Theta(n)}}$ .

**Proposition 14.** *For the function  $\beta$  from Theorem 1, one can set  $\beta(n) = 2^{4^{n+o(n)}}$ .*

*Proof.* The proof of Theorem 1 shows that one can set  $\beta(n) = n(\psi(n)\theta(n) + 1)$ . Here  $\theta$  is the function from Lemma 9, while  $\psi(n) = \omega(k, n)$ , with  $k = \lceil 2^{n-1}n/\nu(n) \rceil$ , where  $\omega$  and  $\nu$  are the functions from Corollary 6 and Theorem 12, respectively.

Let us first give an upper bound on  $\theta(n)$ . The proof of Lemma 9 shows that we can set  $\theta(n) = \varphi(n-1) + 1$ , where  $\varphi$  is the function from Lemma 8. Since one can choose  $\varphi(n) = n^{3n}$  (see [6, Remark (2)]), we can set  $\theta(n) = (n-1)^{3(n-1)} + 1 = O(n^{3n})$ .

We now consider the function  $\psi(n) = \omega(k, n)$ . As pointed out in Section 2, one can set  $\omega(k, n) = O(kn^{3/2})$ . We need an upper bound on  $k = \lceil 2^{n-1}n/\nu(n) \rceil$ , i.e., a lower bound on  $\nu(n)$ . The proof of Theorem 12 shows that one can choose  $\nu(n) = \frac{2^n}{\sigma(n)^n n!}$ , where  $\sigma$  is the function from Theorem 11. As shown in [15, Theorem 4], one can set  $\sigma(n) = 8n \cdot 15^{2^{2n+1}}$ . Therefore we can choose  $\psi(n) = O\left(15^{n \cdot 2^{2n+1}} 8^n n^{n+5/2} n!\right)$ .

Overall, we can set  $\beta(n) = n(\psi(n)\theta(n) + 1) = O\left(15^{n \cdot 2^{2n+1}} 8^n n^{4n+7/2} n!\right) = 2^{4^{n+o(n)}}.$   $\square$

We now prove that any lower bound on  $\beta(n)$  is doubly exponential in  $n$ .

**Proposition 15.** *Every function  $\beta$  from Theorem 1 satisfies  $\beta(n) \geq 2^{2^{n-2}} - 2$  for all  $n \in \mathbb{N}$ .*

*Proof.* We make use of a construction from [1, Remark 3.10]. The so-called *Sylvester sequence*  $(t_n)_{n \in \mathbb{N}}$  is defined by

$$t_n := \begin{cases} 2 & \text{if } n = 1, \\ (t_{n-1} - 1)t_{n-1} + 1 & \text{otherwise.} \end{cases}$$

In the following we assume  $n \geq 2$ , as otherwise the statement is trivial.

One easily checks by induction that

$$\frac{1}{t_1} + \dots + \frac{1}{t_{n-1}} + \frac{1}{t_n - 1} = 1.$$

It follows that the simplex

$$T := \text{conv}(\{\mathbf{0}, t_1 e^1, \dots, t_{n-1} e^{n-1}, (t_n - 1) e^n\}) \subseteq \mathbb{R}^n$$

is lattice-free (actually, even maximal lattice-free; see [1]). We modify the simplex  $T$  using an appropriate perturbation of its vertex  $(t_n - 1)e^n$ . E.g., we can consider the simplex

$$Q := \text{conv}(\{\mathbf{0}, t_1 e^1, \dots, t_{n-1} e^{n-1}, p_\varepsilon\}),$$

where  $p_\varepsilon$  is a rational point in  $\text{int}(T)$ ,  $\|p_\varepsilon - (t_n - 1)e^n\| \leq \varepsilon$  and  $\varepsilon > 0$ . By construction, the integer hull of  $Q$  is the facet

$$P := \text{conv}(\{\mathbf{0}, t_1 e^1, \dots, t_{n-1} e^{n-1}\}) \subseteq \mathbb{R}^n$$

of  $T$ . Clearly,  $P$  is an integral polytope of dimension  $n-1$  with the integer point  $(1, \dots, 1, 0) \in \mathbb{R}^n$  in its relative interior. For a small  $\varepsilon$  (say,  $\varepsilon < 1$ ) the width of  $Q$  in direction  $e^n$  is greater than  $t_n - 2$ . Applying Lemma 13 in the same manner as in the proof of Theorem 2, we obtain  $Q^{(i)} \neq P$  for  $i = t_n - 2$ . Since, as shown in [13, page 1026],  $t_n \geq 2^{2^{n-2}}$  for every  $n \in \mathbb{N}$ , we have  $\beta(n) \geq t_n - 2 \geq 2^{2^{n-2}} - 2$ .  $\square$

As one can see from our proofs, the doubly exponential behavior of the minimal  $\beta(n)$  is inherited from the doubly exponential behavior of  $\sigma(n)$  from Theorem 11. On the other hand, the choice of an upper bound for  $\omega(n)$  and  $\varphi(n)$  is not that relevant: we arrive at the assertion of Proposition 14 as long as  $\omega(n)$  and  $\varphi(n)$  are chosen to be of order  $2^{2^{o(n)}}$  (that is, very weak bounds for  $\omega(n)$  and  $\varphi(n)$  would suffice).

## 6 NP-completeness of AFF-IHULL

Let AFF-IHULL denote the following decision problem: given  $x \in \mathbb{Q}^n$  and a rational polyhedron  $Q \subseteq \mathbb{R}^n$  (described by a system of linear inequalities with rational coefficients), decide whether  $x \in \text{aff}(Q_I)$ .

**Proposition 16.** *AFF-IHULL is NP-complete.*

*Proof.* We first show that  $\text{AFF-IHULL} \in \text{NP}$ . Let  $(x, Q)$  be an arbitrary instance of problem AFF-IHULL. Let  $d$  be the dimension of  $Q$ . Using [21, Theorem 17.1] one can show the existence of vectors  $z^0, \dots, z^d \in \mathbb{Q}^n$  whose description size is polynomial in the description size of  $Q$  and such that  $\text{aff}(Q_I) = \text{aff}(\{z^0, \dots, z^d\})$ . It is known that there exists a polynomial algorithm which verifies, for given  $x, z^0, \dots, z^d \in \mathbb{Q}^n$ , whether  $x \in \text{aff}(\{z^0, \dots, z^d\})$  (in fact, the latter condition is reduced to solving a system of linear equalities). Thus the sequence  $z^0, \dots, z^d$  can be taken as the certificate, and we get  $\text{AFF-IHULL} \in \text{NP}$ .

We conclude the proof by showing that 3SAT is polynomially reducible to AFF-IHULL. Let  $\phi$  be an arbitrary 3CNF formula with  $n$  variables  $x_1, \dots, x_n$  and  $m$  clauses. It is well-known that satisfiability of  $\phi$  can be expressed by a system of linear inequalities. We interpret  $x_1, \dots, x_n$  as 0/1-variables. Each clause from  $\phi$  generates a linear inequality. E.g., if  $\phi$  contains the clause  $x_1 \wedge x_2 \wedge \overline{x_3}$ , then we introduce the inequality  $x_1 + x_2 + (1 - x_3) \geq 1$  (we proceed similarly for other possible clauses). In this way we construct inequalities  $a^j x \geq b_j$  with  $j \in \{1, \dots, m\}$ , where  $a^j \in \mathbb{Z}^n$  and  $b_j \in \mathbb{Z}$ . From the fact that each clause uses at most three variables we get:

$$a^j x \in \{-3, \dots, 3\} \quad \forall x \in \{0, 1\}^n \quad \forall j \in \{1, \dots, m\}, \quad (1)$$

$$b_j \in \{-2, \dots, 1\} \quad \forall j \in \{1, \dots, m\}. \quad (2)$$

We introduce an additional variable  $x_{n+1}$  and define the following system for

variables  $x_1, \dots, x_{n+1}$ :

$$a^j x \geq b_j - 4(1 - x_{n+1}) \quad \forall j \in \{1, \dots, m\}, \quad (3)$$

$$0 \leq x_i \leq 3 - 2x_{n+1} \quad \forall i \in \{1, \dots, n\}, \quad (4)$$

$$0 \leq x_{n+1}, \quad (5)$$

$$x_i \in \mathbb{Z} \quad \forall i \in \{1, \dots, n+1\}. \quad (6)$$

Note that in (3)  $x$  denotes the vector  $(x_1, \dots, x_n)$  (that is, the variable  $x_{n+1}$  is not included as a component). For the above system the following conditions can be verified in a straightforward way.

- (a) Every solution of the system satisfies  $x_{n+1} \in \{0, 1\}$  (see (4), (5) and (6)).
- (b) Each element of  $\{0, 1\}^n \times \{0\}$  is a solution of the system (in view of (1) and (2)).
- (c) Every solution with  $x_{n+1} = 1$  lies in  $\{0, 1\}^n \times \{1\}$  (see (4) and (6)).
- (d) The system has a solution lying in  $\{0, 1\}^n \times \{1\}$  if and only if the 3CNF formula  $\phi$  is satisfiable (see (3)).

Let  $Q$  be the rational polyhedron in  $\mathbb{R}^{n+1}$  defined by (3), (4) and (5). From (a), (b) and (c) we see that  $\text{aff}(Q_I)$  is either  $\mathbb{R}^n \times \{0\}$  or  $\mathbb{R}^{n+1}$ . Thus, using (d),  $\phi$  is satisfiable if and only if  $e^{n+1} \in \text{aff}(Q_I)$ . This shows that 3SAT is polynomially reducible to AFF-IHULL.  $\square$

Since the absolute values of the coefficients in the system (3)–(6) are at most 4, we have actually shown that AFF-IHULL is NP-complete even in the strong sense (i.e., also in the case where the integer values used to define instances of AFF-IHULL are represented in the unary encoding; see, e.g., [9, §4.2]).

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